SB-3590

M. Sc. (Part-II) (Applied Mathematics) Examination

March / April – 2011

AM-201 : Partial Differential Equations

Time : 3 Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fillup strictly the details of ☛ signs on your answer book.	Seat No. :
Name of the Examination : ☛ M. Sc. (Part-2) (APPLIED MATHEMATICS)	
Name of the Subject : ☛ AM-201 : Partial Differential Equations	
☛ Subject Code No. : 3 5 9 0 ☛ Section No. (1, 2,.....) : Nil	Student's Signature

(2) Attempt all questions.

(3) All questions carry equal marks.

(4) Follow usual notations and conventions.

1 **(a)** Consider the system of first order partial differential equations (PDEs) which governs the one dimensional flow of an ideal gas with velocity $v = v(x,t)$, density $\rho = \rho(x,t)$ and pressure $p = p(x,t)$.

$$(\rho v)_x + \rho_t = 0$$

$$v v_x + v_t = -\frac{1}{\rho} p_x$$

$$v p_x + p = -\gamma p v_x$$

where, γ is a physical constant determined by specific heat of gas.

Classify the above system of PDEs as hyperbolic, parabolic or elliptic:

(b) Classify the scalar partial differential equation:

$$u_{x_1x_1} + 3u_{x_1x_2} + 3u_{x_2x_1} + u_{x_2x_2} + u_{x_2x_3} + u_{x_3x_2} + u_{x_3x_3} = 0$$

as hyperbolic, parabolic or elliptic.

1 (c) Explain the following boundary conditions associated with PDEs:

i. Dirichlet condition

ii. Neumann condition,

iii. Robin condition

OR

1 (a) Explain the separation of variables method to solve the Laplace equation

$u_{xx} + u_{yy} = 0.$

(b) Derive the one dimensional wave equation

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where c^2 is a constant.

under the following assumptions:

- The mass of string per unit length is constant. The string is perfectly elastic and does not offer any resistance to bending.
- The tension caused by stretching the string before fixing it at the ends is so large that the action of the gravitational force on the string can be neglected.
- The string performs small transverse motions in a vertical plane.

2 (a) A tightly stretched string with fixed end points $x = 0$ *and* $x = l$ is initially at rest in its equilibrium position. If it is set by vibrating by giving to each of its points a velocity $\lambda x(l - x)$ find the displacement of string at any distance x from one end at any time t.

(b) Solve the heat equation $u_t = u_{xx}\,; x > 0, t > 0$ by Fourier transforms, subject to the conditions:

(i) $u = 0$ *when* $x = 0, t > 0$

(ii) $u = \begin{cases} 1\,; 0 < x < 1 \\ 0 \quad , x \geq 1 \end{cases}$ *when* $t = 0$

(iii) $u(x, t)$ is bounded.

OR

2) (a) An insulated rod of length 20 c.m. has its ends A and B maintained at 30°C and 80°C respectively until steady state condition prevails. If the change consists of raising the temperature of A to 40°C and reducing that of B to 60°C Find the temperature at a distance x from A at time t.

(b) A rectangular Plate with insulated surface is 10c.m. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge y = 0 is given by,

$$u = \begin{cases} 20x, & for\ 0 \le x \le 5 \\ 20(10 - x), & for\ 5 < x \le 10 \end{cases}$$

Also, the two long edges $x = 0, x = 10$ as well as the other short edges are kept at 0 °C.

Prove that the temperature u at any point(x,y) is given by

$$u = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{10} e^{\frac{-(2n-1)\pi y}{10}}$$

3 (a) Solve the following example using method of characterisation:

$xu_x + u_y = y; u(x, 0) = x^2$ on the curve $\Gamma = \{(r, 0)\}$.

(b) Define weak solution for partial differential equation. If u is a strong or classical solution of $u_t + (f(u))_x = 0\ x \in R$ with $u(x, 0) = \emptyset(x)$ then prove that it is also a weak solution.

OR

3 (a) Solve: $(p^2 + q^2)y = qz$ using Charpit's method.

(b) Find the solution of $u_t + uu_x = 0 \; with \; u(x,0) = \emptyset(x)$ where

$$\emptyset(x) = \begin{cases} 1, 0 < x < 1 \\ 0, othrwise. \end{cases}$$

in terms of rarefaction and shock wave form.

4 **(a)** Derive the radially symmetric function for Laplace equation in 'n' dimension.

(b) Assume $u \in c_1^2(U_T) \cap c(\overline{U_T})$ solve the heat equation in U_T, then prove that

$\max_{\overline{U_T}} u = \max_{\Gamma_T} u.$

OR

4 **(a)** Define Harmonic Function. If $u \in c^2(u)$ is harmonic then prove that

$u(x) = \int_{\partial B(x,r)} u \, ds = \int_{B(x,r)} u \, dy$ for each ball $B(x,r) \subset U$.

(b) Suppose $u \in c_1^2(U_T)$ solve the heat equation in U_T then prove that

$u \in c^\infty(U_T)$.

5 **(a)** Let $u \in c_1^2(U_T)$ solve the heat equation. Then prove that

$u(x,t) = \frac{1}{4r^n} \int \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$.

(b) If $u \in C(u)$ satisfies the mean value property for each ball $B(x,r) \subset U$ then

prove that $u \in C^\infty(U)$.

OR

5 **(a)** If $\in C(\Gamma_T), f \in C(U_T)$, then prove that there exists at most one solution

$u \in c_1^2(U_T) \cap C(\overline{U_T})$ of the boundary value problem:

$u_t - \Delta u = f \; in \; U_T$

$u = g \; on \; \Gamma_T$.

(b) State and Prove the Lax-Milgram lemma to find weak solution of elliptic

PDEs.
